

SPECTRAL LIFTING IN BANACH ALGEBRAS AND INTERPOLATION IN SEVERAL VARIABLES

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ABSTRACT. Let $\mathcal{A}$ be a unital Banach algebra and let J be a closed two-sided ideal of $\mathcal{A}$. We prove that if any invertible element of $\mathcal{A}/J$ has an invertible lifting in $\mathcal{A}$, then the quotient homomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A}/J$ is a spectral interpolant. This result is used to obtain a noncommutative multivariable analogue of the spectral commutant lifting theorem of Bercovici, Foiaş, and Tannenbaum. This yields spectral versions of Sarason, Nevanlinna–Pick, and Carathéodory type interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra F_n^∞ and $B(\mathcal{K})$, the algebra of bounded operators on a finite dimensional Hilbert space $\mathcal{K}$. A spectral tangential commutant lifting theorem in several variables is considered and used to obtain a spectral tangential version of the Nevanlinna–Pick interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$.

In particular, we obtain interpolation theorems for matrix-valued bounded analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant and not the norm.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{D}$ denote the unit disc in the complex plane, let $z_1, \dots, z_k \in \mathbb{D}$ be given distinct points, and $F_1, \dots, F_k$ be complex $m \times m$ matrices. The classical Nevanlinna–Pick problem [N], [P] consists in finding necessary and sufficient conditions for the existence of an analytic $m \times m$ matrix-valued function $F(z)$ with $F(z_j) = F_j$ ($1 \leq j \leq k$) and such that $\|F\|_\infty \leq 1$.

Motivated by problems in control engineering, such as the design of feedback control systems in the presence of parameter uncertainty, Bercovici, Foiaş, and Tannenbaum proved in [BFT] a spectral generalization of the commutant lifting theorem [SzF1], and obtained a spectral version of the Nevanlinna–Pick problem, in which the infinity norm is replaced by

$$\rho(F) := \sup\{\|F(z)\|_{\text{sp}} : z \in \mathbb{D}\}$$

($\|A\|_{\text{sp}}$ denotes the spectral radius of an operator A).

The tangential Nevanlinna–Pick problem considered by Fedcina [F] is to find $F \in H^\infty(\mathbb{D}) \otimes \mathbb{C}^m$ with $F(z_j)u_j = v_j$, $j = 1, \dots, k$, and $\|F\|_\infty \leq 1$, where $z_j \in \mathbb{D}$ and $u_j, v_j \in \mathbb{C}^m$ are prescribed. The spectral tangential Nevanlinna–Pick interpolation problem, considered by Bercovici and Foiaş [BF], is to find such an F for which $\rho(F) < 1$. This type of interpolation was also motivated by certain control engineering applications.

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In this paper we find noncommutative multivariable analogues of the above-mentioned results obtained by Bercovici, Foiaş, and Tannenbaum (see [BFT] and [BF]) for the noncommutative analytic Toeplitz algebra F_n^∞ . In particular, we obtain interpolation results (see Corollary 3.7 and Corollary 4.3) for matrix-valued bounded analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant and not the norm.

We expect these results to play a role in multivariable control and systems theory, as it does in the case $n = 1$. We mention the papers [BV] and [B] for recent results in multivariable linear systems.

We need to recall some facts concerning the noncommutative analytic Toeplitz algebra F_n^∞ and its connection with the function theory on the open unit ball of $\mathbb{C}^n$. Let $F^2(H_n) = \mathbb{C}1 \oplus \bigoplus_{m \geq 1} H_n^{\otimes m}$ be the full Fock space on n generators, where H_n is an n -dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, \dots, e_n\}$ if n is finite, and $\{e_1, e_2, \dots\}$ if $n = \infty$. For each $i = 1, 2, \dots$, define the left creation operator by $S_i \xi := e_i \otimes \xi$, $\xi \in F^2(H_n)$.

We shall denote by $\mathcal{P}$ the set of all $p \in F^2(H_n)$ which are finite sums of tensor monomials. Define F_n^∞ as the set of all $g \in F^2(H_n)$ such that

$$\|g\|_\infty := \sup\{\|g \otimes p\|_{F^2(H_n)} : p \in \mathcal{P}, \|p\|_{F^2(H_n)} \leq 1\} < \infty.$$

We denote by $\mathcal{A}_n$ the closure of $\mathcal{P}$ in $(F_n^\infty, \|\cdot\|_\infty)$. The Banach algebra F_n^∞ (resp. $\mathcal{A}_n$) can be viewed as a noncommutative analogue of the Hardy space $H^\infty(\mathbb{D})$ (resp. disc algebra $A(\mathbb{D})$); when $n = 1$ they coincide.

In [Po7, Theorem 3.1] we proved that $\mathcal{A}_n$ is completely isometrically isomorphic to the norm-closed algebra generated by any sequence $V_1, \dots, V_n$ of isometries with $V_1 V_1^* + \dots + V_n V_n^* \leq I$, and the identity. It follows from [Po5, Theorem 4.3] that the noncommutative analytic Toeplitz algebra F_n^∞ can be identified with the WOT-closed algebra generated by the left creation operators $S_1, \dots, S_n$, and the identity. The algebras F_n^∞ and $\mathcal{A}_n$ were introduced by the author in [Po3] in connection with a noncommutative von Neumann inequality, and have been studied in several papers [Po2], [Po5], [Po6], [Po7], [Po9], [ArPo1], and recently in [DP1], [DP2], [ArPo2], [DP3], and [Po8].

We established a strong connection between the algebra F_n^∞ and the function theory on the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ through the noncommutative von Neumann inequality [Po3] (see also [Po5], [Po7], and [Po9]). In particular, we proved that there is a completely contractive homomorphism

$$\Phi : F_n^\infty \rightarrow H^\infty(\mathbb{B}_n), \quad f(S_1, \dots, S_n) \mapsto f(\lambda_1, \dots, \lambda_n),$$

where $(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$. A characterization of the analytic functions in the range of the map Φ was obtained in [ArPo2] and [DP3]. W. Arveson proved that Φ is not surjective [Arv] and the functions in its range are the multipliers of a certain function Hilbert space. In [ArPo2], [DP3], it was proved that $F_n^\infty / \ker \Phi$ is an operator algebra which can be identified with $\mathcal{W}_n^\infty := P_{F_s^2} F_n^\infty|_{F_s^2}$, the compression to the symmetric Fock space $F_s^2 \subseteq F^2(H_n)$. In [Po8], [Po9], [Arv], [ArPo2], [DP3], [AMc], and [BTV], a good case is made that the appropriate commutative multivariable analogue of $H^\infty(\mathbb{D})$ is the algebra $\mathcal{W}_n^\infty$, which is the WOT-closed algebra generated by $B_i := P_{F_s^2} S_i|_{F_s^2}$, $i = 1, \dots, n$, and the identity. In this paper, we provide further evidence that F_n^∞ (resp. $\mathcal{W}_n^\infty$) is a noncommutative (resp. commutative) multivariate analogue of $H^\infty(\mathbb{D})$.

Let $\mathcal{A}$ be a unital Banach algebra and denote by $\text{Inv}(\mathcal{A})$ the group of invertible elements of $\mathcal{A}$. Given $a \in \mathcal{A}$, we define the $\mathcal{A}$ -spectral radius of a by setting

$$\rho_{\mathcal{A}}(a) := \inf\{\|xax^{-1}\| : x \in \text{Inv}(\mathcal{A})\}.$$

Since the spectral radius of $a \in \mathcal{A}$ is $\|a\|_{\text{sp}} = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$, it is clear that $\|a\|_{\text{sp}} = \|xax^{-1}\|_{\text{sp}}$ for any $x \in \text{Inv}(\mathcal{A})$. Now, it is easy to see that

$$\|a\|_{\text{sp}} \leq \rho_{\mathcal{A}}(a) \leq \|a\|,$$

for any $a \in \mathcal{A}$. Note that if $\mathcal{A} = B(\mathcal{H})$ (or $\mathcal{A}$ is any C^* -subalgebra of $B(\mathcal{H})$) then $\|a\|_{\text{sp}} = \rho_{\mathcal{A}}(a)$ (see [R]). There are some other examples of Banach algebras such that $\|a\|_{\text{sp}} = \rho_{\mathcal{A}}(a)$ for any $a \in \mathcal{A}$. It was proved in [BFT] that this equality holds if $\mathcal{A}$ is the commutant of an isometry (resp. normal operator) on a Hilbert space.

Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras, and $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital contractive homomorphism. We say that Φ is a quotient interpolant if

$$\|b\| = \inf\{\|a\| : a \in \mathcal{A}, \Phi(a) = b\}$$

for any $b \in \mathcal{B}$. We say that $b \in \mathcal{B}$ with $\rho_{\mathcal{B}}(b) < 1$ has a spectral lifting if there exists $a \in \mathcal{A}$ such that $\Phi(a) = b$ and $\rho_{\mathcal{A}}(a) < 1$. The homomorphism Φ is called a spectral interpolant if any $b \in \mathcal{B}$ has a spectral lifting.

Problem. Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital contractive homomorphism which is also a quotient interpolant. When is Φ a spectral interpolant ?

We show, in Section 2, that this problem has a positive answer if $\text{Inv}(\mathcal{B}) \subseteq \Phi(\text{Inv}(\mathcal{A}))$. This relation holds, for example, if the group of invertible elements of $\mathcal{B}$ is connected (in particular, if $\mathcal{B}$ is finite dimensional or equal to $B(\mathcal{H})$).

The results of Section 2 are used in Section 3 to obtain a noncommutative multivariable analogue (see Theorem 3.1) of the spectral commutant lifting theorem of Bercovici-Foiaş-Tannenbaum. This yields spectral versions of Sarason ([S]), Nevanlinna-Pick, and Carathéodory type interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra F_n^∞ and $B(\mathcal{K})$, the algebra of bounded operators on a finite dimensional Hilbert space $\mathcal{K}$.

In Section 4, we obtain a spectral tangential commutant lifting theorem in several variables (see Theorem 4.1). This leads to a spectral tangential Nevanlinna-Pick interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$ (see Theorem 4.2).

Problems concerning the optimal solutions to these spectral interpolation problems in several variables, and explicit algorithm for finding the optimal interpolants will be considered in a future paper.

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2. SPECTRAL LIFTING IN BANACH ALGEBRAS

The notation and definitions from Section 1 are used throughout the paper. Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital contractive homomorphism. We call Φ a norm preserving interpolant if for any $b \in \mathcal{B}$ there exists $a \in \mathcal{A}$ such that $\Phi(a) = b$ and $\|a\| = \|b\|$. Notice that any norm preserving interpolant is a quotient interpolant. Examples of norm preserving interpolants will be presented in Section 3.

Theorem 2.1. *Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital contractive homomorphism with the property that $\text{Inv}(\mathcal{B}) \subseteq \Phi(\text{Inv}(\mathcal{A}))$ and*

$$\|b\| = \inf\{\|a\| : a \in \mathcal{A}, \Phi(a) = b\}$$

for any $b \in \mathcal{B}$. Then

$$(2.1) \quad \rho_{\mathcal{B}}(b) = \inf\{\rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \Phi(a) = b\}$$

for any $b \in \mathcal{B}$. In particular, Φ is a spectral interpolant.

Proof. Let $b \in \mathcal{B}$ and $a \in \mathcal{A}$ with $\Phi(a) = b$. Since Φ is a contractive homomorphism and $\Phi(\text{Inv}(\mathcal{A})) \subseteq \text{Inv}(\mathcal{B})$ we have

$$\begin{aligned} \rho_{\mathcal{A}}(a) &= \inf\{\|waw^{-1}\| : w \in \text{Inv}(\mathcal{A})\} \\ &\geq \inf\{\|\Phi(waw^{-1})\| : w \in \text{Inv}(\mathcal{A})\} \\ &= \inf\{\|\Phi(w)b\Phi(w)^{-1}\| : w \in \text{Inv}(\mathcal{A})\} \\ &\geq \inf\{\|zbz^{-1}\| : z \in \text{Inv}(\mathcal{B})\} \\ &= \rho_{\mathcal{B}}(b). \end{aligned}$$

Therefore,

$$\rho_{\mathcal{B}}(b) \leq \inf\{\rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \Phi(a) = b\}.$$

Now, let $\epsilon > 0$ and choose $z \in \text{Inv}(\mathcal{B})$ such that

$$(2.2) \quad \|zbz^{-1}\| \leq \rho_{\mathcal{B}}(b) + \frac{\epsilon}{2}.$$

Since $zbz^{-1} \in \mathcal{B}$, according to the hypothesis, for any $\epsilon > 0$, there exists $d \in \mathcal{A}$ such that

$$(2.3) \quad \Phi(d) = zbz^{-1} \quad \text{and} \quad \|d\| \leq \|zbz^{-1}\| + \frac{\epsilon}{2}.$$

Since $\Phi(\text{Inv}(\mathcal{A})) \supseteq \text{Inv}(\mathcal{B})$, we find $w \in \text{Inv}(\mathcal{A})$ such that $\Phi(w) = z$. Notice that $y := w^{-1}dw \in \mathcal{A}$ and

$$\Phi(y) = \Phi(w)^{-1}\Phi(d)\Phi(w) = z^{-1}(zbz^{-1})z = b.$$

Now, using (2.2) and (2.3), we infer that

$$\rho_{\mathcal{A}}(y) \leq \|wyw^{-1}\| = \|d\| \leq \|zbz^{-1}\| + \frac{\epsilon}{2} \leq \rho_{\mathcal{B}}(b) + \epsilon.$$

Therefore,

$$\rho_{\mathcal{B}}(b) \geq \inf\{\rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \Phi(a) = b\}.$$

Using relation (2.1), it is easy to see that if $b \in \mathcal{B}$, then $\rho_{\mathcal{B}}(b) < 1$ if and only if there exists $a \in \mathcal{A}$ such that $\Phi(a) = b$ and $\rho_{\mathcal{A}}(a) < 1$. This completes the proof. $\blacksquare$

Corollary 2.2. *Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras such that the group $\text{Inv}(\mathcal{B})$ is connected. Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital contractive homomorphism which is also a quotient interpolant. Then Φ is a spectral interpolant.*

Proof. Let us prove that

$$(2.4) \quad \Phi(\text{Inv}(\mathcal{A})) = \text{Inv}(\mathcal{B}).$$

The inclusion $\Phi(\text{Inv}(\mathcal{A})) \subseteq \text{Inv}(\mathcal{B})$ is clear. Conversely, let $x \in \text{Inv}(\mathcal{B})$. Since $\text{Inv}(\mathcal{B})$ is connected, it is well known that

$$x = \exp(z_1) \cdots \exp(z_k)$$

for some $z_1, \dots, z_k \in \mathcal{B}$. Due to the hypothesis, there exist $w_1, \dots, w_k \in \mathcal{A}$ such that $\Phi(w_i) = z_i$, $i = 1, \dots, k$. Denote $y := \exp(w_1) \cdots \exp(w_k) \in \text{Inv}(\mathcal{A})$ and notice that $\Phi(y) = \exp(\Phi(w_1)) \cdots \exp(\Phi(w_k)) = x$. Hence $\Phi(\text{Inv}(\mathcal{A})) \supseteq \text{Inv}(\mathcal{B})$ and (2.4) holds. ■

Remark 2.3. If $\mathcal{B}$ is a finite dimensional algebra, then $\text{Inv}(\mathcal{B}) = \exp(\mathcal{B})$, hence $\text{Inv}(\mathcal{B})$ is connected.

Corollary 2.4. *Let $\mathcal{A}$ be a unital Banach algebra and let J be a closed two-sided ideal of $\mathcal{A}$. If any invertible element of $\mathcal{A}/J$ has an invertible lifting in $\mathcal{A}$, then the quotient homomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A}/J$ is a spectral interpolant, i.e., $\rho_{\mathcal{A}/J}(a+J) < 1$ if and only if there exists $b \in a + J$ such that $\rho_{\mathcal{A}}(b) < 1$.*

Proof. Apply Theorem 2.1 to the quotient homomorphism Φ . ■

Let us remark that, in general, there are invertible elements in $\mathcal{A}/J$ which can not be lifted to invertible elements in $\mathcal{A}$. For example, if $\pi : B(H^2) \rightarrow B(H^2)/K(H^2)$ is the quotient homomorphism into the Calkin algebra, and S is the unilateral shift on the Hardy space H^2 , then $\pi(S)$ is invertible and there is no invertible operator $T \in B(H^2)$ such that $\pi(T) = \pi(S)$.

An important particular case, when Corollary 2.4 can be applied, is when the quotient algebra $\mathcal{A}/J$ is finite dimensional. Applications of this result will be considered in the next section.

3. NONCOMMUTATIVE SPECTRAL COMMUTANT LIFTING AND INTERPOLATION

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $s_1, \dots, s_n$, and let e be its neutral element. For any $\sigma := s_{i_1} \cdots s_{i_k} \in \mathbb{F}_n^+$ we define its length $|\sigma| := k$, and $|e| = 0$. On the other hand, if $T_i \in B(\mathcal{H})$, $i = 1, \dots, n$, we denote $T_\sigma := T_{i_1} \cdots T_{i_k}$ and $T_e := I_{\mathcal{H}}$.

Let us recall from [Po1], [Po2], and [Po4] some results concerning the non-commutative dilation theory for n -tuples of operators. A sequence of operators $\mathcal{T} := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, $i = 1, \dots, n$, is called contractive (or row contraction) if $T_1 T_1^* + \cdots + T_n T_n^* \leq I_{\mathcal{H}}$. We say that a sequence of isometries $\mathcal{V} := [V_1, \dots, V_n]$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is a minimal isometric dilation of $\mathcal{T}$ if the following properties are satisfied:

- (i) $V_1 V_1^* + \cdots + V_n V_n^* \leq I_{\mathcal{K}}$;
- (ii) $V_i^*|_{\mathcal{H}} = T_i^*$, $i = 1, \dots, n$;
- (iii) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

The minimal isometric dilation of $\mathcal{T}$ is uniquely determined up to an isomorphism. We need to recall the noncommutative commutant lifting theorem [Po4] (see [SzF1], [SzF2], [DMP] for the classical case).

Let $\mathcal{T} := [T_1, \dots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \dots, V_n]$ be its minimal isometric dilation on a Hilbert $\mathcal{K} \supseteq \mathcal{H}$. If $X \in B(\mathcal{H})$ and $XT_i = T_i X$ for any $i = 1, \dots, n$, then there exists $X_\infty \in B(\mathcal{K})$ satisfying the following properties:

- (i) $X_\infty V_i = V_i X_\infty$, for any $i = 1, \dots, n$;
- (ii) $X_\infty^*|_{\mathcal{H}} = X^*$;
- (iii) $\|X_\infty\| = \|X\|$.

Let $\mathcal{T} := [T_1, \dots, T_n]$ be a row contraction with $T_i \in B(\mathcal{H})$ and let $\mathcal{V} := [V_1, \dots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Let $X \in \{T_1, \dots, T_n\}'$, and denote

$$\text{Dil}(X) := \{Y \in \{V_1, \dots, V_n\}' : P_{\mathcal{H}}Y = XP_{\mathcal{H}}\},$$

where $P_{\mathcal{H}}$ is the orthogonal projection on $\mathcal{H}$. According to the noncommutative commutant lifting, we have $\text{Dil}(X) \neq \emptyset$.

In what follows we obtain a noncommutative multivariable analogue of the spectral commutant lifting theorem of Bercovici-Foiaş-Tannenbaum [BFT].

Theorem 3.1. *Let $\mathcal{T} := [T_1, \dots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \dots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $\mathcal{H}$ is finite dimensional and $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \dots, V_n\}$, then*

$$\rho_{\{T_1, \dots, T_n\}'}(X) = \inf\{\rho_{\{V_1, \dots, V_n\}'}(Y) : Y \in \text{Dil}(X)\}$$

for any $X \in \{T_1, \dots, T_n\}'$.

Proof. Let $\Phi : \{V_1, \dots, V_n\}' \rightarrow \{T_1, \dots, T_n\}'$ be defined by $\Phi(Y) := P_{\mathcal{H}}Y|_{\mathcal{H}}$. Since $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \dots, V_n\}$, we have $Y^*(\mathcal{H}) \subseteq \mathcal{H}$ for any $Y \in \{V_1, \dots, V_n\}'$. Since $\mathcal{V} := [V_1, \dots, V_n]$ is the minimal isometric dilation of $\mathcal{T}$, we have $V_i^*|_{\mathcal{H}} = T_i^*, i = 1, \dots, n$. Now, it is easy to see that

$$(P_{\mathcal{H}}Y|_{\mathcal{H}})T_i = T_i(P_{\mathcal{H}}Y|_{\mathcal{H}}) \quad \text{for any } i = 1, 2, \dots, n.$$

Therefore, the mapping Φ is well-defined. On the other hand, since $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \dots, V_n\}$, we infer that Φ is a unital contractive homomorphism, and $\Phi(Y) = X$ is equivalent to $P_{\mathcal{H}}Y = XP_{\mathcal{H}}$. According to the noncommutative commutant lifting theorem, for any $X \in \{T_1, \dots, T_n\}'$ there exists $Y \in \{V_1, \dots, V_n\}'$ such that $P_{\mathcal{H}}Y = XP_{\mathcal{H}}$ and $\|Y\| = \|X\|$. Therefore, Φ is a norm preserving interpolant. Since $\mathcal{H}$ is finite dimensional, the algebra $\{T_1, \dots, T_n\}'$ is finite dimensional. Applying Theorem 2.1 and Remark 2.3, in the particular case when $\mathcal{A} := \{V_1, \dots, V_n\}'$ and $\mathcal{B} := \{T_1, \dots, T_n\}'$, the result follows. ■

Corollary 3.2. *Let $\mathcal{T} := [T_1, \dots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \dots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $\mathcal{H}$ is finite dimensional and $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \dots, V_n\}$, then, given $X \in \{T_1, \dots, T_n\}'$, $\rho_{\{T_1, \dots, T_n\}'}(X) < 1$ if and only if there exists $Y \in \text{Dil}(X)$ such that $\rho_{\{V_1, \dots, V_n\}'}(Y) < 1$.*

In what follows, we use the noncommutative spectral commutant lifting theorem to obtain spectral versions of Sarason, Nevanlinna-Pick, and Carathéodory type interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra F_n^∞ and $B(\mathcal{K})$. In particular, we obtain interpolation results for matrix-valued analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant.

According to Theorem 1.2 from [Po6], the commutant of F_n^∞ , which we denote by R_n^∞ , is equal to $U^*F_n^\infty U$, where U is the unitary operator on $F^2(H_n)$ defined by $U(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}) = e_{i_k} \otimes \dots \otimes e_{i_2} \otimes e_{i_1}$. Moreover, the commutant of R_n^∞ is equal to F_n^∞ .

A complete description of the invariant subspace structure of F_n^∞ was obtained in [Po2, Theorem 2.2] (even in a more general setting). A subspace $\mathcal{N}$ of $F^2(H_n)$ is invariant under $S_1, \dots, S_n$ if and only if $\mathcal{N} = \bigoplus_{\lambda \in \Lambda} U^* \varphi_\lambda U[F^2(H_n)]$, for some

family $\{\varphi_\lambda \in F_n^\infty : \lambda \in \Lambda\}$ of isometries with orthogonal ranges (see also [Po6] and [DP1]). Let us remark that $\mathcal{M} \subseteq F^2(H_n)$ is hyperinvariant for $\{S_1, \dots, S_n\}$, i.e., invariant for $\{S_1, \dots, S_n\}'$, if and only if $U\mathcal{M}$ is invariant for $\{S_1, \dots, S_n\}$.

Theorem 3.3. *Let $\mathcal{K}$ be a finite dimensional Hilbert space and let $\mathcal{N} \subseteq F^2(H_n)$ be a finite dimensional subspace with the property that $\mathcal{N}$ and $U\mathcal{N}$ are invariant under $S_1^*, \dots, S_n^*$. Then $X \in B(\mathcal{N} \otimes \mathcal{K})$ commutes with each $P_{\mathcal{N}}S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}, i = 1, \dots, n$, and*

$$\rho_{P_{\mathcal{N}}R_n^\infty|_{\mathcal{N}} \bar{\otimes} B(\mathcal{K})}(X) < 1$$

if and only there exists $\Psi \in R_n^\infty \bar{\otimes} B(\mathcal{K})$ such that

$$P_{\mathcal{N} \otimes \mathcal{K}}\Psi = XP_{\mathcal{N} \otimes \mathcal{K}} \text{ and } \rho_{R_n^\infty \bar{\otimes} B(\mathcal{K})}(\Psi) < 1.$$

Proof. According to [Po8], we have

$$\mathcal{B} := \{P_{\mathcal{N}}S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}, i = 1, \dots, n\}' = P_{\mathcal{N} \otimes \mathcal{K}}(R_n^\infty \bar{\otimes} B(\mathcal{K}))|_{\mathcal{N} \otimes \mathcal{K}}.$$

Notice that $\mathcal{B}$ is a finite dimensional algebra. Let $\mathcal{A} := R_n^\infty \bar{\otimes} B(\mathcal{K})$ and define $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ by $\Phi(Y) = P_{\mathcal{N} \otimes \mathcal{K}}Y|_{\mathcal{N} \otimes \mathcal{K}}$. Since $S_i^*(U\mathcal{N}) \subseteq U\mathcal{N}$ for any $i = 1, \dots, n$, and $\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}' = R_n^\infty \bar{\otimes} B(\mathcal{K})$, it is easy to see that $[F^2(H_n) \otimes \mathcal{K}] \ominus [\mathcal{N} \otimes \mathcal{K}]$ is hyperinvariant for $\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}$ and the mapping Φ is a unital contractive homomorphism. Since $\mathcal{N}$ is invariant under $S_1^*, \dots, S_n^*$, it is clear that the operator matrix $[P_{\mathcal{N}}S_1|_{\mathcal{N}}, \dots, P_{\mathcal{N}}S_n|_{\mathcal{N}}]$ is a C_0 -row contraction and its minimal isometric dilation is $[S_1, \dots, S_n]$ (see [Po1]). Therefore, the minimal isometric dilation of $[P_{\mathcal{N}}S_1|_{\mathcal{N}} \otimes I_{\mathcal{K}}, \dots, P_{\mathcal{N}}S_n|_{\mathcal{N}} \otimes I_{\mathcal{K}}]$ is $[S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}]$. According to the noncommutative commutant lifting theorem, for any $X \in \mathcal{B}$ there exists $\Psi \in R_n^\infty \bar{\otimes} B(\mathcal{K})$, such that $P_{\mathcal{N} \otimes \mathcal{K}}\Psi = XP_{\mathcal{N} \otimes \mathcal{K}}$ and $\|X\| = \|\Psi\|$. Therefore, $\Phi(\Psi) = X$ and Φ is a norm preserving interpolant. Applying Corollary 3.2, the result follows. ■

Notice that the element Ψ in Theorem 3.3 satisfies $\|\Psi\|_{\text{sp}} \leq \rho_{R_n^\infty \bar{\otimes} B(\mathcal{K})}(\Psi) < 1$. It would be nice to know if $\rho_{R_n^\infty \bar{\otimes} B(\mathcal{K})}(\Psi) = \|\Psi\|_{\text{sp}}$ for any $\Psi \in R_n^\infty \bar{\otimes} B(\mathcal{K})$. This equality holds if $n = 1$ (see [BFT]).

Let us remark that the finite dimensionality hypothesis can be dropped in Theorem 3.3 for those subspaces $\mathcal{N}$ and $\mathcal{K}$ for which one can prove that any invertible element $f \in P_{\mathcal{N}}R_n^\infty|_{\mathcal{N}} \bar{\otimes} B(\mathcal{K})$ can be lifted to an invertible element $g \in R_n^\infty \bar{\otimes} B(\mathcal{K})$, i.e., $P_{\mathcal{N} \otimes \mathcal{K}}g|_{\mathcal{N} \otimes \mathcal{K}} = f$. We do not have yet any nontrivial example when this lifting property holds and $\mathcal{N}, \mathcal{K}$ are infinite dimensional.

Let J be a WOT-closed, two-sided ideal of F_n^∞ and define $J(1) := \{\Psi(1) : \Psi \in J\}$ and $\mathcal{N}_J := F^2(H_n) \ominus J(1)$. Let us remark that $\mathcal{N}_J$ and $U\mathcal{N}_J$ are invariant subspaces under $S_i^*, i = 1, \dots, n$, therefore, Theorem 3.3 works in the case when $\dim \mathcal{N}_J < \infty$.

Corollary 3.4. *Let $\mathcal{K}$ be a finite dimensional Hilbert space and let J be a WOT-closed two-sided ideal of F_n^∞ such that $\dim \mathcal{N}_J < \infty$. Then the quotient homomorphism*

$$\Phi : F_n^\infty \bar{\otimes} B(\mathcal{K}) \rightarrow F_n^\infty \bar{\otimes} B(\mathcal{K}) / (J \bar{\otimes} B(\mathcal{K}))$$

is a spectral interpolant.

Proof. According to [ArPo2], the quotient algebra $F_n^\infty \bar{\otimes} B(\mathcal{K}) / (J \bar{\otimes} B(\mathcal{K}))$ is completely isometrically isomorphic to $P_{\mathcal{N}_J}F_n^\infty|_{\mathcal{N}_J} \bar{\otimes} B(\mathcal{K})$, which is finite dimensional. Using Theorem 3.3, we infer that Φ is a spectral interpolant. The proof is complete. ■

It will be interesting to see if this result remains true if $\mathcal{N}_J$ is infinite dimensional (at least for some particular cases, if not in general). The obstruction in the infinite dimensional case seems to be the lifting of the invertible elements of a quotient algebra $\mathcal{A}/J$ to invertible elements of $\mathcal{A}$ (see Section 2 for an example). In the finite dimensional case, Corollary 3.4 leads to our spectral interpolation results for F_n^∞ (see Theorem 3.6 and Theorem 3.8).

Let $F_s^2(H_n)$ be the symmetric Fock space and $\mathcal{W}_n^\infty$ be the WOT-closed algebra generated by $B_i := P_{F_s^2(H_n)} S_i|_{F_s^2(H_n)}$, $i = 1, \dots, n$, and the identity. This algebra has been studied in [Po9], [Arv], [ArPo2], [DP3]. The following theorem can be seen as a spectral version of Sarason's interpolation theorem for $H^\infty(\mathbb{D})$ (see [S]), in a commutative and multivariable setting.

Theorem 3.5. *Let $\mathcal{E} \subseteq F_s^2(H_n)$ be a finite dimensional invariant subspace under $B_1^*, \dots, B_n^*$ and let $\mathcal{K}$ be a finite dimensional Hilbert space. Then $f \in B(\mathcal{E} \otimes \mathcal{K})$ commutes with each $P_{\mathcal{E}} B_i|_{\mathcal{E} \otimes \mathcal{K}}$, $i = 1, \dots, n$, and*

$$\rho_{P_{\mathcal{E} \otimes \mathcal{K}}(\mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K}))|_{\mathcal{E} \otimes \mathcal{K}}}(f) < 1$$

if and only if there exists $g \in \mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K})$ such that

$$P_{\mathcal{E} \otimes \mathcal{K}} g|_{\mathcal{E} \otimes \mathcal{K}} = f \quad \text{and} \quad \rho_{\mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K})}(g) < 1.$$

Proof. Since $F_s^2(H_n)$ is invariant under each S_i^* , $i = 1, \dots, n$, it is easy to see that $\mathcal{E}$ has the same property. Taking into account that $\mathcal{W}_n^\infty$ is the compression of F_n^∞ to the symmetric Fock space, one can see that f commutes with $P_{\mathcal{E} \otimes \mathcal{K}}(S_i \otimes I_{\mathcal{K}})|_{\mathcal{E} \otimes \mathcal{K}}$. As in the proof of Theorem 3.3, using the noncommutative commutant lifting theorem, we find $\phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that $P_{\mathcal{E} \otimes \mathcal{K}}(U^* \otimes I_{\mathcal{K}})\phi(U \otimes I)|_{\mathcal{E} \otimes \mathcal{K}} = f$ and $\|f\| = \|\phi\|$. Hence, $P_{\mathcal{E} \otimes \mathcal{K}}\phi|_{\mathcal{E} \otimes \mathcal{K}} = f$. Setting $g := P_{F_s^2(H_n) \otimes \mathcal{K}}\phi|_{F_s^2(H_n) \otimes \mathcal{K}} \in \mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K})$, we have $P_{\mathcal{E} \otimes \mathcal{K}}g|_{\mathcal{E} \otimes \mathcal{K}} = f$ and $\|f\| \leq \|g\| \leq \|\phi\| = \|f\|$. This shows that $\|f\| = \|g\|$. Define $\mathcal{A} := \mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K})$, $\mathcal{B} := P_{\mathcal{E} \otimes \mathcal{K}}(\mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K}))|_{\mathcal{E} \otimes \mathcal{K}}$ and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be defined by $\Phi(g) := P_{\mathcal{E} \otimes \mathcal{K}}(g)|_{\mathcal{E} \otimes \mathcal{K}}$. We just proved that Φ is a unital contractive homomorphism and also a norm preserving interpolant. Now, the result follows by applying the results of Section 2 in our setting. ■

Let us remark that a result similar to Corollary 3.4 holds for the algebra $\mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K})$.

In what follows we obtain a spectral version of Nevanlinna-Pick interpolation for the noncommutative analytic Toeplitz algebra F_n^∞ (see [ArPo2], [DP3], and [Po8]). As mentioned in the first section, there exists a unital contractive homomorphism

$$\Psi : F_n^\infty \bar{\otimes} B(\mathcal{K}) \rightarrow H^\infty(\mathbb{B}_n) \bar{\otimes} B(\mathcal{K})$$

defined by $[\Psi(f)](\lambda) := f(\lambda)$, $\lambda \in \mathbb{B}_n$.

Theorem 3.6. *Let $\mathcal{K}$ be a finite dimensional Hilbert space, $W_j \in B(\mathcal{K})$, and λ_j , $j = 1, \dots, k$, be distinct elements in $\mathbb{B}_n$. Then there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that*

$$\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < 1 \quad \text{and} \quad \Phi(\lambda_j) = W_j, \quad j = 1, \dots, k,$$

if and only if there exist invertible operators $M_j \in B(\mathcal{K})$, $j = 1, \dots, k$, such that

$$(3.1) \quad \left[\frac{I_{\mathcal{K}} - (M_i W_i M_i^{-1})(M_j W_j M_j^{-1})^*}{1 - \langle \lambda_i, \lambda_j \rangle} \right]_{1 \leq i, j \leq k} > 0.$$

Proof. Let $\lambda_j := (\lambda_{j1}, \dots, \lambda_{jn}) \in \mathbb{B}_n$, $j = 1, \dots, k$. For any $\alpha := s_{j_1} s_{j_2} \dots s_{j_m}$ in $\mathbb{F}_n^+$, let $\lambda_{j\alpha} := \lambda_{jj_1} \lambda_{jj_2} \dots \lambda_{jj_m}$ and $\lambda_e := 1$. Define $z_{\lambda_j} \in F^2(H_n)$ by setting

$$z_{\lambda_j} := \sum_{\alpha \in \mathbb{F}_n^+} \bar{\lambda}_{j\alpha} e_\alpha, \quad j = 1, 2, \dots, k.$$

Let $\mathcal{N} := \text{span}\{z_{\lambda_j} : j = 1, \dots, k\}$ and $X \in B(\mathcal{N} \otimes \mathcal{K})$ be defined by

$$(3.2) \quad X^*(z_{\lambda_j} \otimes h) := z_{\lambda_j} \otimes W_j^* h, \quad h \in \mathcal{K}.$$

Notice that $S_i^* z_{\lambda_j} = \bar{\lambda}_{ji} z_{\lambda_j}$ for any $i = 1, \dots, n$; $j = 1, \dots, k$. Hence, the subspaces $\mathcal{N}$ and $U\mathcal{N}$ are invariant under each S_i^* , $i = 1, \dots, n$. Define $T_i \in B(\mathcal{N} \otimes \mathcal{K})$ by $T_i := P_{\mathcal{N}} S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}$. Since $z_{\lambda_1}, \dots, z_{\lambda_k}$ are linearly independent, the operator $X \in B(\mathcal{N} \otimes \mathcal{K})$ given by (3.2) is well defined.

Notice that $XT_i = T_i X$ for any $i = 1, \dots, k$. Indeed,

$$\begin{aligned} T_i^* X^*(z_{\lambda_j} \otimes h) &= T_i^*(z_{\lambda_j} \otimes W_j^* h) = S_i^* z_{\lambda_j} \otimes W_j^* h \\ &= \bar{\lambda}_{ji} z_{\lambda_j} \otimes W_j^* h \end{aligned}$$

and

$$X^* T_i^*(z_{\lambda_j} \otimes h) = X^*(\bar{\lambda}_{ji} z_{\lambda_j} \otimes h) = \bar{\lambda}_{ji} z_{\lambda_j} \otimes W_j^* h.$$

Applying Theorem 3.3, we infer that

$$(3.3) \quad \rho_{\{T_1, \dots, T_n\}'}(X) < 1$$

if and only there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that

$$(3.4) \quad P_{\mathcal{N} \otimes \mathcal{K}}(U^* \otimes I)\Phi(U \otimes I) = X P_{\mathcal{N} \otimes \mathcal{K}} \quad \text{and} \quad \rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < 1.$$

Since $[F^2(H_n) \otimes \mathcal{K}] \ominus [\mathcal{N} \otimes \mathcal{K}]$ is hyperinvariant for $\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}$, the first relation in (3.4) is equivalent to

$$(3.5) \quad P_{\mathcal{N} \otimes \mathcal{K}}(U^* \otimes I)\Phi(U \otimes I)|_{\mathcal{N} \otimes \mathcal{K}} = X.$$

Since $U(z_{\lambda_j}) = z_{\lambda_j}$, $j = 1, \dots, k$, and $\langle \phi, z_{\lambda_i} \rangle = \phi(\lambda_i)$ for any $\phi := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ in $F^2(H_n)$, it is easy to see that

$$\begin{aligned} \langle (U^* \otimes I)\Phi(U \otimes I)(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle \\ &= \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle \Phi(\lambda_j)h, h' \rangle = \langle X(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle \\ &= \langle \Phi(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle = \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle W_j h, h' \rangle. \end{aligned}$$

for any $j = 1, \dots, k$, and $h, h' \in \mathcal{K}$. This shows that (3.5) holds if and only if $\Phi(\lambda_j) = W_j$ for any $j = 1, \dots, k$. Notice that relation (3.3) holds if and only if there exists $M \in \text{Inv}(\{T_1, \dots, T_n\}')$ such that $\|MXM^{-1}\| < 1$. It is easy to see that $M^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes M_j^* h$, $h \in \mathcal{K}$, for some invertible operators $M_j \in B(\mathcal{K})$, $j = 1, \dots, k$. On the other hand, notice that

$$M^{*-1} X^* M^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes (M_j W_j M_j^{-1})^* h$$

and $\|MXM^{-1}\| < 1$ is equivalent to $I_{\mathcal{N} \otimes \mathcal{K}} - (MXM^{-1})(MXM^{-1})^* > 0$, which is equivalent to (3.1). This completes the proof. $\blacksquare$

Let us remark that the inequality (3.1) can be replaced with

$$(3.6) \quad \rho_{P_{\mathcal{N}} F_n^\infty|_{\mathcal{N}} \bar{\otimes} B(\mathcal{K})}(X) < 1.$$

In the particular case when $n = 1$, we find again Theorem 4 from [BFT]. As mentioned in [BFT], since $P_{\mathcal{N}}F_n^\infty|_{\mathcal{N}}\bar{\otimes}B(\mathcal{K})$ is finite dimensional, conditions of type (3.6) can be checked using computer algorithms.

Corollary 3.7. *Let $\mathcal{K}$ be a finite dimensional Hilbert space, $W_j \in B(\mathcal{K})$, and λ_j , $j = 1, \dots, k$, be distinct elements in $\mathbb{B}_n$. If there exist invertible operators $M_j \in B(\mathcal{K})$, $j = 1, \dots, k$, such that*

$$\left[\frac{I_{\mathcal{K}} - (M_i W_i M_i^{-1})(M_j W_j M_j^{-1})^*}{1 - \langle \lambda_i, \lambda_j \rangle} \right]_{1 \leq i, j \leq k} > 0,$$

then there exists $f \in H^\infty(\mathbb{B}_n) \bar{\otimes} B(\mathcal{K})$ such that

$$f(\lambda_j) = W_j, \quad j = 1, \dots, k, \quad \text{and} \quad \sup_{\lambda \in \mathbb{B}_n} \|f(\lambda)\|_{\text{sp}} < 1.$$

Proof. Using Theorem 3.6, we find $f \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that $f(\lambda_j) = W_j$, $i = 1, \dots, k$, and $\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(f) < 1$. As in the proof of Theorem 2.1, we infer that

$$\|\Psi(f)\|_{\text{sp}} \leq \rho_{H^\infty(\mathbb{B}_n) \bar{\otimes} B(\mathcal{K})}(\Psi(f)) \leq \rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(f) < 1.$$

On the other hand, similarly to [BFT, Proposition 3], one can prove that

$$\|\Psi(f)\|_{\text{sp}} = \sup_{\lambda \in \mathbb{B}_n} \|f(\lambda)\|_{\text{sp}}.$$

This completes the proof. ■

Let $\mathcal{P}_m$ be the set of all polynomials in $F^2(H_n)$ of degree $\leq m$, and let $\mathcal{P}_m^\infty := \{p(S_1, \dots, S_n) : p \in \mathcal{P}_m\}$. Let $J_{>m}^\infty$ be the WOT-closed two-sided ideal of F_n^∞ generated by $\{S_\alpha : \alpha \in \mathbb{F}_n^+, |\alpha| = m+1\}$. The following result is a spectral version of the noncommutative Carathéodory interpolation problem for F_n^∞ (see [Po6] and [Po8]).

Theorem 3.8. *Let $\mathcal{K}$ be a finite dimensional Hilbert space and let $p \in \mathcal{P}_m^\infty \bar{\otimes} B(\mathcal{K})$. Then there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ with*

$$\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < 1$$

such that $\Phi = p + g$ for some $g \in J_{>m}^\infty \bar{\otimes} B(\mathcal{K})$ if and only if

$$(3.7) \quad \rho_{\mathcal{C}}[P_{\mathcal{P}_m \otimes \mathcal{K}}(U^* \otimes I)p(U \otimes I)|_{\mathcal{P}_m \otimes \mathcal{K}}] < 1$$

where $\mathcal{C} := P_{\mathcal{P}_m \otimes \mathcal{K}}(F_n^\infty \bar{\otimes} B(\mathcal{K}))|_{\mathcal{P}_m \otimes \mathcal{K}}$.

Proof. Let $\mathcal{N} := \mathcal{P}_m$ and $X := P_{\mathcal{P}_m \otimes \mathcal{K}}(U^* \otimes I)p(U \otimes I)|_{\mathcal{P}_m \otimes \mathcal{K}}$. Notice that X commutes with each $P_{\mathcal{P}_m} S_i|_{\mathcal{P}_m} \otimes I_{\mathcal{K}}$, $i = 1, \dots, n$, and $\mathcal{P}_m = U\mathcal{P}_m$ is invariant under each $S_1^*, \dots, S_n^*$. According to Theorem 3.3, relation (3.7) holds if and only if there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ with $P_{\mathcal{P}_m \otimes \mathcal{K}}(U^* \otimes I)\Phi(U \otimes I) = XP_{\mathcal{P}_m \otimes \mathcal{K}}$ and $\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < 1$. Hence, we infer that

$$(3.8) \quad P_{\mathcal{P}_m \otimes \mathcal{K}}(U^* \otimes I)(\Phi - p)(U \otimes I)|_{\mathcal{P}_m \otimes \mathcal{K}} = 0.$$

On the other hand, every element $f \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ has a unique Fourier expansion $f \sim \sum_{\alpha \in \mathbb{F}_n^+} S_\alpha \otimes W_{(\alpha)}$ determined by

$$f(1 \otimes h) = \sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes W_{(\alpha)} h \in F^2(H_n) \otimes \mathcal{K},$$

where $W_{(\alpha)} \in B(\mathcal{K})$ are given by $\langle W_{(\alpha)}h, k \rangle = \langle f(1 \otimes h), e_\alpha \otimes k \rangle$ for any $h, k \in \mathcal{K}$, and $\alpha \in \mathbb{F}_n^+$ (see [Po8]). Using now relation (3.8), one can easily see that $g := \Phi - p \in J_{>m}^\infty \bar{\otimes} B(\mathcal{K})$. This completes the proof. $\blacksquare$

Using Theorem 3.5, one can obtain a version of Theorem 3.8 for the algebra $\mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K})$, in a similar manner. We leave this task to the reader.

4. SPECTRAL TANGENTIAL COMMUTANT LIFTING IN SEVERAL VARIABLES

Let $\mathcal{T} := [T_1, \dots, T_n]$ be a row contraction with $T_i \in B(\mathcal{H})$, and $\mathcal{V} := [V_1, \dots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Let $\mathcal{M} \subseteq \mathcal{H}$ be an invariant subspace under each T_i^* , $i = 1, \dots, n$, and $X \in B(\mathcal{H})$ be such that $X\mathcal{H} \subseteq \mathcal{M}$ and

$$(4.1) \quad (P_{\mathcal{M}}T_i|_{\mathcal{M}})X = XT_i, \quad \text{for any } i = 1, \dots, n.$$

According to the noncommutative commutant lifting theorem, there exists $Y \in \{V_1, \dots, V_n\}'$ with $P_{\mathcal{M}}Y = XP_{\mathcal{H}}$. Define

$$\text{Dil}_{\mathcal{M}}(X) := \{Y \in \{V_1, \dots, V_n\}' : P_{\mathcal{M}}Y = XP_{\mathcal{H}}\}$$

and

$$\rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) := \inf\{\|P_{Z^*\mathcal{M}}Z^{-1}XZ\| : Z \in \text{Inv}(\{T_1, \dots, T_n\}')\}.$$

Notice that if $\mathcal{M} = \mathcal{H}$, then $\rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) = \rho_{\{T_1, \dots, T_n\}'}(X)$.

In what follows we extend the spectral tangential commutant lifting theorem of Bercovici and Foias [BF] to our noncommutative multivariable setting.

Theorem 4.1. *Let $\mathcal{T} := [T_1, \dots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \dots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $\mathcal{H}$ is finite dimensional, $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \dots, V_n\}$, and $\mathcal{M} \subseteq \mathcal{H}$ is an invariant subspace under each T_i^* , $i = 1, \dots, n$, then, for every $X \in B(\mathcal{H})$ such that $X\mathcal{H} \subseteq \mathcal{M}$ and $(P_{\mathcal{M}}T_i|_{\mathcal{M}})X = XT_i$, $i = 1, \dots, n$, we have*

$$(4.2) \quad \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) = \inf\{\rho_{\{V_1, \dots, V_n\}'}(Y) : Y \in \text{Dil}_{\mathcal{M}}(X)\}.$$

Proof. Denote the right hand side of (4.2) by t . Let $\epsilon > 0$ and choose $Y \in \text{Dil}_{\mathcal{M}}(X)$ such that $\rho_{\{V_1, \dots, V_n\}'}(Y) < t + \epsilon$. Hence, there is $W \in \text{Inv}(\{V_1, \dots, V_n\}')$ such that $\|W^{-1}YW\| < t + \epsilon$. Since $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \dots, V_n\}$, we infer that $P_{\mathcal{H}}WP_{\mathcal{H}} = P_{\mathcal{H}}W$. Let $Z := P_{\mathcal{H}}W|_{\mathcal{H}}$ and notice that $Z \in \text{Inv}(\{T_1, \dots, T_n\}')$ and

$$(4.3) \quad Z^{-1} = P_{\mathcal{H}}W^{-1}|_{\mathcal{H}}.$$

The subspace $\mathcal{M}_* := Z^*\mathcal{M}$ is invariant under each T_i^* , $i = 1, \dots, n$, and satisfies $\mathcal{M}_* = \mathcal{H} \ominus Z^{-1}(\mathcal{H} \ominus \mathcal{M})$. Hence, we deduce the relations

$$(4.4) \quad P_{\mathcal{M}_*}Z^{-1} = P_{\mathcal{M}_*}Z^{-1}P_{\mathcal{M}} \quad \text{and} \quad P_{\mathcal{M}}Z = P_{\mathcal{M}}ZP_{\mathcal{M}_*}.$$

Since $Y \in \text{Dil}_{\mathcal{M}}(X)$ and $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \dots, V_n\}$, we can use (4.4) and (4.3) to infer that

$$\begin{aligned} \|P_{\mathcal{M}_*}Z^{-1}XZ\| &= \|P_{\mathcal{M}_*}Z^{-1}(P_{\mathcal{M}}Y|_{\mathcal{H}})Z\| = \|P_{\mathcal{M}_*}Z^{-1}(P_{\mathcal{H}}Y|_{\mathcal{H}})Z\| \\ &= \|P_{\mathcal{M}_*}(P_{\mathcal{H}}W^{-1}|_{\mathcal{H}})(P_{\mathcal{H}}Y|_{\mathcal{H}})(P_{\mathcal{H}}W|_{\mathcal{H}})\| \leq \|P_{\mathcal{H}}(W^{-1}YW)|_{\mathcal{H}}\| \\ &\leq \|W^{-1}YW\| < t + \epsilon. \end{aligned}$$

Since $\epsilon > 0$, we deduce that $\rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) \leq t$.

Now, let us prove the converse. Let $\epsilon > 0$ and choose $Z \in \text{Inv}(\{T_1, \dots, T_n\}')$ such that

$$(4.5) \quad \|P_{\mathcal{M}_*} Z^{-1} X Z\| \leq \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon.$$

Since $\{T_1, \dots, T_n\}'$ is finite dimensional, we use Theorem 2.1 and Remark 2.3 when $\Phi : \{V_1, \dots, V_n\}' \rightarrow \{T_1, \dots, T_n\}'$ and $\Phi(W) = P_{\mathcal{H}} W|_{\mathcal{H}}$, to find $W \in \text{Inv}(\{V_1, \dots, V_n\}')$ such that $Z = P_{\mathcal{H}} W|_{\mathcal{H}}$. Denote $X_* := P_{\mathcal{M}_*} Z^{-1} X Z$ and notice that

$$(4.6) \quad (P_{\mathcal{M}_*} T_i|_{\mathcal{M}_*}) X_* = X_* T_i, \quad i = 1, \dots, n.$$

Indeed, since $\mathcal{M}_*$ is invariant under each T_i^* , $i = 1, \dots, n$, we have $P_{\mathcal{M}_*} T_i P_{\mathcal{M}_*} = P_{\mathcal{M}_*} T_i$, $i = 1, \dots, n$. Using this relation together with (4.1) and (4.4), we infer that, for any $i = 1, \dots, n$,

$$\begin{aligned} X_* T_i &= P_{\mathcal{M}_*} Z^{-1} X Z T_i = P_{\mathcal{M}_*} Z^{-1} X T_i Z \\ &= P_{\mathcal{M}_*} Z^{-1} (P_{\mathcal{M}} T_i|_{\mathcal{M}}) X Z = P_{\mathcal{M}_*} Z^{-1} T_i X Z \\ &= P_{\mathcal{M}_*} T_i Z^{-1} X Z = P_{\mathcal{M}_*} T_i P_{\mathcal{M}_*} Z^{-1} X Z \\ &= P_{\mathcal{M}_*} T_i X_*. \end{aligned}$$

According to (4.6), the noncommutative commutant lifting theorem, and relation (4.5), we find $Y_* \in \text{Dil}_{\mathcal{M}_*}(X_*)$ satisfying

$$(4.7) \quad \|Y_*\| = \|X_*\| \leq \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon.$$

Set $Y := W Y_* W^{-1}$ and let us show that $Y \in \text{Dil}_{\mathcal{M}}(X)$. Notice that

$$(4.8) \quad X = P_{\mathcal{M}} Z X_* Z^{-1}.$$

Indeed, using (4.4), we have

$$\begin{aligned} P_{\mathcal{M}} Z X_* Z^{-1} &= P_{\mathcal{M}} Z (P_{\mathcal{M}_*} Z^{-1} X Z) Z^{-1} = P_{\mathcal{M}} Z P_{\mathcal{M}_*} Z^{-1} X \\ &= P_{\mathcal{M}} Z Z^{-1} X = P_{\mathcal{M}} X = X. \end{aligned}$$

Since $P_{\mathcal{M}_*} Y_* = X_* P_{\mathcal{H}}$, $Z^{-1} = P_{\mathcal{H}} W^{-1}|_{\mathcal{H}}$, and $Y(\mathcal{K} \ominus \mathcal{H}) \subseteq \mathcal{K} \ominus \mathcal{H}$, we can use relation (4.8) to obtain

$$\begin{aligned} X P_{\mathcal{H}} &= P_{\mathcal{M}} Z X_* Z^{-1} P_{\mathcal{H}} = P_{\mathcal{M}} Z P_{\mathcal{M}_*} Y_* Z^{-1} P_{\mathcal{H}} \\ &= P_{\mathcal{M}} Z P_{\mathcal{H}} Y_* Z^{-1} P_{\mathcal{H}} = P_{\mathcal{M}} (P_{\mathcal{H}} Z|_{\mathcal{H}}) (P_{\mathcal{H}} Y_*|_{\mathcal{H}}) (P_{\mathcal{H}} W^{-1}|_{\mathcal{H}}) P_{\mathcal{H}} \\ &= P_{\mathcal{M}} (P_{\mathcal{H}} W Y_* W^{-1}|_{\mathcal{H}}) P_{\mathcal{H}} = P_{\mathcal{M}} Y P_{\mathcal{H}} = P_{\mathcal{M}} Y. \end{aligned}$$

According to (4.7), we have $\|W^{-1} Y W\| = \|Y_*\| \leq \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon$. Hence $\rho_{\{V_1, \dots, V_n\}'}(Y) \leq \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon$ and $t \leq \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon$. This completes the proof. $\blacksquare$

The following result is a spectral version of the tangential Nevanlinna-Pick interpolation problem for F_n^∞ (see [Po8]).

Theorem 4.2. *Let λ_j , $j = 1, \dots, k$, be distinct elements in $\mathbb{B}_n$ and let $\mathcal{K}$ be a finite dimensional Hilbert space. If $u_1, \dots, u_k, v_1, \dots, v_k \in \mathcal{K}$ with $u_i \neq 0$, $j = 1, \dots, k$, and $\delta > 0$, then there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that*

$$\Phi(\lambda_j)^* u_j = v_j, \quad j = 1, \dots, k, \quad \text{and} \quad \rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < \delta$$

if and only if there exist invertible operators $Z_j \in B(\mathcal{K})$, $j = 1, \dots, k$, such that

$$(4.9) \quad \left[\frac{\langle \delta Z_j u_j, \delta Z_i u_i \rangle - \langle Z_j v_j, Z_i v_i \rangle}{1 - \langle \lambda_j, \lambda_i \rangle} \right]_{1 \leq i, j \leq k} > 0.$$

Proof. Let $\mathcal{N} := \text{span}\{z_{\lambda_j} : j = 1, \dots, k\}$ and $\mathcal{M} := \mathbb{C}z_{\lambda_1} \otimes u_1 + \dots + \mathbb{C}z_{\lambda_k} \otimes u_k$ be a subspace of $\mathcal{N} \otimes \mathcal{K}$. Define $X(\{\lambda_j\}, \{u_j\}, \{v_j\}) \in B(\mathcal{N} \otimes \mathcal{K}, \mathcal{M})$ by setting $X(\{\lambda_j\}, \{u_j\}, \{v_j\})^*(z_{\lambda_j} \otimes u_j) := z_{\lambda_j} \otimes v_j$, $j = 1, \dots, k$. For each $i = 1, \dots, n$, define $T_i := P_{\mathcal{N}} S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}$ and notice that $T_i^* X^* = X^* T_i^*|_{\mathcal{M}}$, where $X := X(\{\lambda_j\}, \{u_j\}, \{v_j\})$. Hence, $X T_i = P_{\mathcal{M}} T_i X$ for any $i = 1, \dots, n$.

As in the proof of Theorem 3.3, the minimal isometric dilation of the sequence $[T_1, \dots, T_n]$ is $[S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}]$ and $[F^2(H_n) \otimes \mathcal{K}] \ominus [\mathcal{N} \otimes \mathcal{K}]$ is hyperinvariant for $\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}$. Since $\mathcal{M} \subseteq \mathcal{N} \otimes \mathcal{K}$ is invariant under each T_i^* , $i = 1, \dots, n$, we can apply Theorem 4.1 and infer that

$$\rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) = \inf\{\rho_{\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}'}(Y) : Y \in \text{Dil}_{\mathcal{M}}(X)\}.$$

Since $\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}' = U^* F_n^\infty U \bar{\otimes} B(\mathcal{K})$, we can see that

$$(4.10) \quad \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) < \delta$$

if and only if there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that $\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < \delta$ and

$$(4.11) \quad P_{\mathcal{M}}(U^* \otimes I)\Phi(U \otimes I) = X P_{\mathcal{N} \otimes \mathcal{K}}.$$

Notice that

$$\begin{aligned} \langle P_{\mathcal{M}}(U^* \otimes I)\Phi(U \otimes I)(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j \rangle &= \langle \Phi(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j \rangle \\ &= \langle z_{\lambda_i}, z_{\lambda_j} \rangle \langle \Phi(\lambda_j)k, u_j \rangle \\ &= \langle z_{\lambda_i}, z_{\lambda_j} \rangle \langle k, \Phi(\lambda_j)^* u_j \rangle \end{aligned}$$

and $\langle X(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j \rangle = \langle z_{\lambda_i}, z_{\lambda_j} \rangle \langle k, v_j \rangle$ for any $k \in \mathcal{K}$ and $i, j = 1, \dots, k$. Therefore, the relation (4.11) holds if and only if $\Phi(\lambda_j)^* u_j = v_j$, $j = 1, \dots, k$. On the other hand, if $Z \in \{T_1, \dots, T_n\}'$ then

$$(4.12) \quad Z^*(z_{\lambda_j} \otimes k) = z_{\lambda_j} \otimes Z_j k, \quad k \in \mathcal{K},$$

for some $Z_j \in B(\mathcal{K})$, $j = 1, \dots, k$. Notice that Z is invertible if and only if Z_j is invertible for any $j = 1, \dots, k$. Moreover, using the definition of $X = X(\{\lambda_j\}, \{u_j\}, \{v_j\})$ and (4.12), we have

$$Z^* X^* (\{\lambda_j\}, \{u_j\}, \{v_j\}) Z^{*-1}|_{Z^* \mathcal{M}} = X^* (\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}).$$

Therefore,

$$\rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) = \inf\{\|X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\})\| : Z_j \in B(\mathcal{K}) \text{ are invertible}\}$$

and relation (4.10) holds if and only if there exist invertible operators $Z_j \in B(\mathcal{K})$ such that $\|X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\})\| < \delta$. This inequality is equivalent to

$$\delta^2 I - X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}) X^* (\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}) > 0,$$

which is equivalent to (4.9). This completes the proof. $\blacksquare$

We remark that (4.9) can be replaced by relation (4.10). As a consequence of Theorem 4.2, when the distinct elements in $\mathbb{B}_n$ are $\bar{\lambda}_j$, $j = 1, \dots, k$, we infer the following spectral tangential interpolation result for matrix-valued bounded analytic functions in the unit ball of $\mathbb{C}^n$.

Corollary 4.3. *Let λ_j , $j = 1, \dots, k$, be distinct elements in $\mathbb{B}_n$ and let $\mathcal{K}$ be a finite dimensional Hilbert space. If $u_1, \dots, u_k, v_1, \dots, v_k \in \mathcal{K}$ with $u_i \neq 0$, $j = 1, \dots, k$, $\delta > 0$, and there exist invertible operators $Z_j \in B(\mathcal{K})$, $j = 1, \dots, k$, such that*

$$\left[\frac{\langle \delta Z_j u_j, \delta Z_i u_i \rangle - \langle Z_j v_j, Z_i v_i \rangle}{1 - \langle \lambda_i, \lambda_j \rangle} \right]_{1 \leq i, j \leq k} > 0,$$

then there exists $F \in H^\infty(\mathbb{B}_n) \bar{\otimes} B(\mathcal{K})$ such that

$$\sup_{\lambda \in \mathbb{B}_n} \|F(\lambda)\|_{sp} < \delta \quad \text{and} \quad F(\lambda_j)u_j = v_j, \quad j = 1, \dots, k.$$

Let us make some remarks on the dependence of $\rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X)$ on the given interpolation data. For each $m = 1, \dots, k$, we define

$$\rho_m := \inf \{ \|X(\{\lambda_j\}_{j=1}^m, \{Z_j u_j\}_{j=1}^m, \{Z_j v_j\}_{j=1}^m)\| : Z_j \in B(\mathcal{K}) \text{ are invertible} \}.$$

A multivariable analogue of [BF, Proposition 4] holds. More precisely, one can prove that if u_k and v_k are linearly independent, then $\rho_{k-1} = \rho_k$. Indeed, suppose that $\rho_{k-1} < \rho_k$. Using Theorem 4.2, we find $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that $\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < \rho_k$ and $\Phi(\lambda_j)^* u_j = v_j$, $j = 1, \dots, k-1$. We may suppose that $\Phi(\lambda_k)^* \notin \mathbb{C}I_{\mathcal{K}}$ because, otherwise, we can replace Φ by $\Phi + \Psi$ for some $\Psi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ satisfying $\Phi(\lambda_j) = 0$, $j = 1, \dots, k-1$, and $\Psi(\lambda_k) \notin \mathbb{C}I_{\mathcal{K}}$. Since we can choose Ψ with very small norm we have $\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi + \Psi) < \rho_k$.

Therefore, since $\Phi(\lambda_k)^* \notin \mathbb{C}I_{\mathcal{K}}$, there exist linearly independent vectors u and v such that $\Phi(\lambda_k)^* u = v$. Since u_k, v_k are linearly independent, we can find $Z_k \in B(\mathcal{K})$ invertible with $Z_k u_k = u$ and $Z_k v_k = v$. Hence, we infer that $\rho_k \leq \rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < \rho_k$, which is a contradiction. Since $\rho_{k-1} \leq \rho_k$, we must have $\rho_{k-1} = \rho_k$. This shows that in Theorem 4.2 we can assume, without loss of generality, that $v_j = \mu_j u_j$, for some $\mu_j \in \mathbb{C}$, $\mu_j \neq 0$, $j = 1, \dots, k$. Similarly to [BF, Proposition 5], one can show that if $k \leq \dim \mathcal{K}$, then

$$\rho_k = \max\{|\mu_1|, \dots, |\mu_k|\}.$$

The case when the number of dependent vector pairs (u_j, v_j) exceeds the dimension of $\mathcal{K}$, and the problem of optimal solutions will be considered in a future paper.

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